Boolean Algebras

P. Danziger

1 Boolean Algebras

We have seen the correspondence between **Sets** with the operations of *union, intersection* and *complement* and **Logic** with the operations *And, Or* and *Not.* We would like to have a way to talk about these systems in a meta way, without reference to the specific operations.
In mathematics such systems are defined by **Axioms**. Axioms are basic rules or truths which the system is assumed to obey. We then can derive Theorems based on the axioms.

Definition 1

A *Booolean Algebra* is a set S together with 2 binary operations on S, denoted $a + b$ and $a \cdot b$ such that for all a, b and $c \in S$

0. **(Closure)**

$$a + b \in S \quad \text{and} \quad a \cdot b \in S$$

1. **(Commutativity)**

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$$

2. **(Associativity)**

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

3. **(Distributivity)**

$$a + (b \cdot c) = (a + b) \cdot (a + c) \quad \text{and} \quad a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

4. **(Identity)** There are distinct elements of S, called **0** and **1** such that

$$a + 0 = a \quad \text{and} \quad (a \cdot 1) = a$$

5. **(Complement)** For every element $a \in S$ there exists an special element $\overline{a} \in S$ such that

$$a + \overline{a} = 1 \quad \text{and} \quad (a \cdot \overline{a}) = 0$$

A given a system which satisfies all the axioms, such as Boolean logic os Sets, is called a *model* for the axioms. Since from these axioms other laws can be proven, it is now only necessary to show that the axioms hold for the model and all other Theorems within the axiomatic system will also hold.

Theorem 2 *For any Boolean Algebra S*

1. **(Uniqueness of Complement)** For any elements a and x of a Boolean Algebra S, if $a+x=1$ and $a \cdot x = 0$ then $x = \overline{a}$.

2. **Uniqueness of 0 and 1** If there exists an element $x \in S$ such that for all $a \in S$ $a+x=a$, then $x=0$.

   If there exists an element $x \in S$ such that for all $a \in S$ $a \cdot x = a$, then $x = 1$.

3. **Double Complement** For any element $a \in S$ $\overline{\overline{a}} = a$.

4. **Idempotent** For all $a \in S$
$$a + a = a \quad \text{and} \quad (a \cdot a) = a.$$

5. **Universal Bound** For every $a \in S$
$$a + 1 = 1 \quad \text{and} \quad (a \cdot 0) = 0.$$

6. **DeMorgan** For any elements a and b of S,
$$\overline{a+b} = \overline{a} \cdot \overline{b} \quad \text{and} \quad \overline{a \cdot b} = \overline{a} + \overline{b}$$

7. **Absorption** For all $a, b \in S$
$$(a+b) \cdot a = a \quad \text{and} \quad (a \cdot b) + a = a$$

8. **Compliments of 0 and 1**
$$\overline{0} = 1 \quad \text{and} \quad \overline{1} = 0$$

Proof of 1 To Prove: For any elements a and x of a Boolean Algebra S, if $a+x=1$ and $a \cdot x = 0$ then $x = \overline{a}$.
Let S be a Boolean Algebra and let $a, x \in S$ be such that, $a + x = 1$ and $a \cdot x = 0$.

$$\begin{array}{llll}
\text{Consider } x & = & x + 0 & \text{(Identity)} \\
 & = & (x + a \cdot \overline{a}) & \text{(Complement)} \\
 & = & (x+a) \cdot (x + \overline{a}) & \text{(Distributivity)} \\
 & = & 1 \cdot (x + \overline{a}) & \text{(Assumption)} \\
 & = & (\overline{a} + a) \cdot (x + \overline{a}) & \text{(Complement)} \\
 & = & \overline{a} + (a \cdot x) & \text{(Distributivity \& Commutativity)} \\
 & = & \overline{a} + 0 & \text{(Assumption)} \\
 & = & \overline{a} & \text{(Identity)}
\end{array}$$

Hence we conclude that $x = \overline{a}$. □